Homework 1 Solution

1. Let $n \in \mathbb{N}^+$ and $\mathbb{F}_2 = \{0, 1\}$ be the binary field, i.e. the finite field of order 2.

   (a) Prove that $M_{n\times n}(\mathbb{F}_2)$, the collection of all $n \times n$ matrices with entries in $\mathbb{F}_2$, is a vector space.

   (b) Prove or disprove that $S = \{A \in M_{n\times n}(\mathbb{F}_2) : \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} = 0\}$ is a subspace of $M_{n\times n}(\mathbb{F}_2)$.

Proof:

(a) For any $A, B, C \in M_{n\times n}(\mathbb{F}_2)$, any $a, b \in \mathbb{F}_2$

- Closed under addition

  $(A+B)_{ij} = A_{ij} + B_{ij} \in \mathbb{F}_2$ Thus $A+B \in M_{n\times n}(\mathbb{F}_2)$

- Closed under scalar multiplication

  $(aA)_{ij} = a \cdot A_{ij} \in \mathbb{F}_2$. Thus $aA \in M_{n\times n}(\mathbb{F}_2)$

- VS 1

  $(A+B)_{ij} = A_{ij} + B_{ij} = B_{ij} + A_{ij} = (B+A)_{ij}$

  Thus $A+B = B+A$

- VS 2

  $((A+B)+C)_{ij} = (A+B)_{ij} + C_{ij} = (A_{ij} + B_{ij}) + C_{ij}$

  $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

  $(A+(B+C))_{ij} = A_{ij} + (B+C)_{ij} = A_{ij} + (B_{ij} + C_{ij})$

  Thus $(A+B)+C = A+(B+C)$

- VS 3

  $\exists\ \vec{0} \in M_{n\times n}(\mathbb{F}_2)$ $\quad \vec{0}_{ij} = 0 \in \mathbb{F}_2$

  s.t. $\quad A + \vec{0} = A$

- VS 4

$\exists A' \in M_{n\times n}(\mathbb{F}_2)$ $\quad A'_{ij} = A_{ij}$

s.t. $(A+A')_{ij} = A_{ij} - A_{ij} = 0$ $\quad$ Thus $A+A' = \vec{0}$

- VS 5

$(1\cdot A)_{ij} = 1\cdot A_{ij} = A_{ij}$ $\quad$ Thus $1\cdot A = A$

- VS 6

$((ab)\cdot A)_{ij} = (ab)\cdot A_{ij} = a\cdot(b\cdot A_{ij}) = a\cdot(bA)_{ij} = (a(bA))_{ij}$

Thus $(ab)\cdot A = a\cdot(b\cdot A)$

- VS 7

$(a(A+B))_{ij} = a\cdot(A+B)_{ij} = a(A_{ij}+B_{ij}) = aA_{ij} + aB_{ij}$

$\qquad (aA+aB)_{ij} = (aA)_{ij} + (aB)_{ij}$

Thus $a(A+B) = aA + aB$

- VS 8

$((a+b)A)_{ij} = (a+b)\cdot A_{ij} = a\cdot A_{ij} + bA_{ij} = (aA)_{ij} + (bA)_{ij}$

$\qquad (aA+bA)_{ij}$

Thus $(a+b)A = aA + bA$

(b)
- The zero matrix $\vec{0} \in S$, since $\sum_i\sum_j 0 = 0$
- $\forall A, B \in S$,

$\sum_i\sum_j (A+B)_{ij} = \sum_i\sum_j A_{ij} + \sum_i\sum_j B_{ij} = 0 + 0 = 0$

Thus $A+B \in S$

- $\forall a \in \mathbb{F}_2$ $\quad A \in S$

$\sum_i\sum_j (aA)_{ij} = \sum_i\sum_j a\cdot A_{ij} = a\cdot \sum_i\sum_j A_{ij} = a\cdot 0 = 0$

Thus $aA \in S$

2. Suppose W_1 and W_2 are two subspaces of V, please give a necessary and sufficient condition such that $W_1 \cup W_2$ is a subspace of V and prove it.

Claim: $W_1 \cup W_2$ is a subspace $\iff W_1 \subset W_2$ or $W_2 \subset W_1$

Proof:

$(\Rightarrow)$

If $W_1 \cup W_2$ is a subspace and $W_1 \not\subset W_2$, we show that $W_2 \subset W_1$.

pick $x_1 \in W_1 \setminus W_2$, for any $x_2 \in W_2$.

Since $x_1, x_2 \in W_1 \cup W_2$. we have $x_1 + x_2 \in W_1 \cup W_2$

Case 1. $x_1 + x_2 \in W_1$.

$\exists\, y \in W_1$ s.t $x_1 + x_2 = y$ i.e. $x_2 = y + (-x_1) \in W_1$

i.e. $W_2 \subset W_1$

Case 2. $x_1 + x_2 \in W_2 \setminus W_1$

$\exists\, y \in W_2 \setminus W_1$ s.t $x_1 + x_2 = y$

i.e. $x_1 = y + (-x_2) \in W_2$ Contradicting to $x_1 \in W_1 \setminus W_2$

Thus $W_2 \subset W_1$

$(\Leftarrow)$ Trivial.

3. Textbook (Friedberg). Sec. 1.3: Q26

26. In $\mathsf{M}_{m\times n}(F)$ define $\mathsf{W}_1 = \{A \in \mathsf{M}_{m\times n}(F) \colon A_{ij} = 0 \text{ whenever } i > j\}$ and $\mathsf{W}_2 = \{A \in \mathsf{M}_{m\times n}(F) \colon A_{ij} = 0 \text{ whenever } i \le j\}$. ($\mathsf{W}_1$ is the set of all upper triangular matrices defined in Exercise 12.) Show that $\mathsf{M}_{m\times n}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2$.

Proof:

① WLOG. let $n \geqslant m$. For any $A \in M_{m\times n}(F)$

Let $A_1 = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_{nn} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ A_{21} & 0 & \cdots & 0 \\ A_{21} & A_{32} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}$

Then $A = A_1 + A_2$, $A_1 \in W_1$ and $A_2 \in W_2$

② $\forall\ B \in W_1 \cap W_2$

Then $B_{ij} = 0 \quad \forall\, i > j$ since $B \in W_1$

and $B_{ij} = 0 \quad \forall\, i \le j$ since $B \in W_2$

i.e. $B_{ij} = 0 \quad \forall\, i, j$. Thus $W_1 \cap W_2 = \{ 0_{m\times n} \}$

By ① and ②. $M_{m\times n}(F) = W_1 \oplus W_2$

4. Textbook (Friedberg). Sec. 1.6: Q31

31. Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) \leq n$.
(b) Prove that $\dim(\mathsf{W}_1 + \mathsf{W}_2) \leq m + n$.

Proof:

(a) Since $W_1 \cap W_2$ is a subspace of W_2 and $\dim(W_2) = n$

By Thm 1-11, $\dim(W_1 \cap W_2) \leq \dim(W_2) = n$

(b) Since W_1 and W_2 are finite-dim

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$
$$\leq m + n$$

5. Textbook (Friedberg). Sec. 1.6: Q33(b)

33. **(a)** Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \varnothing$ and $\beta_1 \cup \beta_2$ is a basis for V.

(b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then $V = W_1 \oplus W_2$.

Proof: Note that $\dim(W_1)$ and $\dim(W_2)$ may not be finite.

(b)

① Since $\beta_1 \cup \beta_2$ is a basis for V.

Thus $V = \text{span}(\beta_1 \cup \beta_2) = \text{span}(\beta_1) + \text{span}(\beta_2) = W_1 + W_2$

② $\forall\, x \in W_1 \cap W_2$.

$\exists\, a_1 \cdots a_m \in F$, $u_1 \cdots u_m \in \beta_1$ s.t. $x = \sum_1^m a_i u_i$

$\exists\, b_1 \cdots b_n \in F$, $v_1 \cdots v_n \in \beta_2$ s.t. $x = \sum_1^n b_j v_j$

Then $0 = x - x = \sum_1^m a_i u_i - \sum_1^n b_j v_j$ ☆

Since $\{u_i\}_1^m \cup \{v_j\}_1^n \subset \beta_1 \cup \beta_2$ and $\beta_1 \cup \beta_2$ lin. ind.

we know that $\{u_i\}_1^m \cup \{v_j\}_1^n$ lin. ind.

Thus ☆ has a unique zero solution $a_i = b_j = 0$

Therefore $x = \vec{0}$, which means $W_1 \cap W_2 = \{\vec{0}\}$

By ① and ②, $V = W_1 \oplus W_2$